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Generalization of the Hardy–Littlewood theorem to the two-index case

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ABSTRACT
In this paper we study methods of summability of series. We extend the results of regular summability methods to the domain of multi-index sequences. In particular, we show the relations between Abel's and Cesáro's methods. The main result is a generalization of the well-known Hardy-Littlewood theorem to double sequences.

KEYWORDS
Summability methods, Hardy-Littlewood theorem, Abel method, Cesáro method

Introduction

Let's recall the definitions of regularity and present methods of summability (limesability) from the 19th and early 20th centuries.

Definition 0.1. *A summability method is called regular if it preserves the value of the sum of a series when this series converges according to the "natural definition of convergence".*

Definition 0.2. *The series $\sum b_n$ is called Abel summable (A) with sum s, if* $\lim\limits_{x\to 1-} \sum b_n x^n = s$

If in the Definition 0.2 the element b_n is replaced by the difference of two consecutive terms of another sequence: $b_n = a_n - a_{n-1}$ then we will obtain the same method for the sequence of partial sums (a_n). Let us recall the definition of Abel summability.

Definition 0.3. *The sequence (a_n) is called Abel summable (A) with sum s, if* $\lim\limits_{x\to 1-} (1-x) \sum a_n x^n = s$.

The next historical summability procedures were the regular Hölder [4] and Cesáro methods.

Definition 0.4. *The sequence (a_n) is called Hölder (H, k) summable with sum s if:*

$$\lim_{n\to\infty} H_n^k = s, \ \textit{where } H_n^k = \frac{H_0^{k-1} + \ldots + H_n^{k-1}}{n+1} \ i \ H_n^0 = a_n.$$

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Definition 0.5. *The sequence (a_n) is called Cesáro (C,k) summable with sum s if:*

$$\lim_{n\to\infty}\frac{A_n^k}{E_n^k}=s,\ \textit{where}\ A_n^k=A_0^{k-1}+\ldots+A_n^{k-1},\ A_n^0=a_n,\ E_n^k=\binom{n+k}{k}.$$

Definition 0.6. *The sequence (a_n) is called Hausdorff (σ,μ_n) summable with sum s, if:*

$$\lim_{m\to\infty}\sum_{n=0}^{m}\binom{m}{n}\left(\Delta^{m-n}\mu_n\right)a_n=s,$$

where

$$\Delta^p\mu_n=\sum_{i=0}^{p}(-1)^i\binom{p}{i}\mu_{n+i}.$$

The Hausdorff methods contain the Cesáro and Hölder methods as special cases.
Let us recall that:

Theorem 0.1. *If the sequence (a_n) is summable by Cesáro's $(C,1)$ method, then it is summable by Abel's method to the same limit.*

Theorem 0.2. (**Hardy – Littlewood**)
If the sequence (a_n) is summable by Abel's method, and $a_n\geq 0$ then it is summable by Cesáro's $(C,1)$ method to the same limit.

The proofs of the above-formulated theorems can be found in the work [3].
We will give the definitions and Toeplitz's theorems for single-index sequences (classical theorem) and multi-index sequences.

Definition 0.7. *A matrix $C=[c_{mn}]$ is regular if it transforms every convergent sequence into a sequence convergent to the same limit, i.e.*

$$\forall a_n(a_n\to\sigma)\Rightarrow(\forall_m\sum_{n=0}^{\infty}c_{mn}a_n\ \textit{is convergent and}\ \sum_{n=0}^{\infty}c_{mn}a_n\to\sigma).$$

In this way, we will define regularity as "$1+2+1$" regularity.

The symbol "1+2+1" indicates that a 2-dimensional matrix acts on a 1-dimensional vector to give a 1-dimensional vector.
Let us now recall the classical theorem of Otto Toeplitz from 1911 giving the conditions for equivalence of regularity.

Theorem 0.3. *The matrix $C=[c_{mn}]$ is "$1+2+1$" regular if and only if the following conditions are satisfied:*

T1) $\forall n\geq 0\ \lim_{m\to\infty}c_{mn}=0$;
T2) $\lim_{m\to\infty}\sum_{n=0}^{\infty}c_{mn}=1$;
T3) $\exists N>0\,\forall m\geq 0\sum_{n=0}^{\infty}|c_{mn}|<M$.

The original proof of the above theorem can be found in [7], see also [1] or [3].

The above Toeplitz theorem has many applications, in particular in the proof of the classical Hausdorff moment theorem, see [1]. This theorem has found applications in a number of analytic and algebraic problems in several variables.

We now give the definition and Toeplitz's theorem for the case of a multi-index sequence.

Definition 0.8. *Let's assume* $C = [c_{m_1,\dots,m_{r_2},n_1,\dots,n_{r_1}}]$*, where* r_1, r_2 *are fixed natural numbers. We call the matrix* C *„*$r_1 + (r_1 + r_2) + r_2$*" regular if for every* r_1*-indicator sequence* $(a_{n_1,\dots,n_{r_1}})$ *convergent to* σ*, the* r_2*-indicator sequence*

$$V = \left[v_{m_1,\dots,m_{r_2}}\right], \text{ where } v_{m_1,\dots,m_{r_2}} = \sum_{\substack{n_1=0\\ \vdots \\ n_{r_1}=0}}^{\infty} a_{n_1,\dots,n_{r_1}} c_{m_1,\dots,m_{r_2},n_1,\dots,n_{r_1}},$$

tends to σ*, where* $m_1, m_2, \dots, m_{r_2}$ *tend to infinity.*

Theorem 0.4. *The matrix* C *is "*$r_1+(r_1+r_2)+r2$*" regular if and only if the following conditions hold:*

T1)

$$\forall n_1, \dots, n_{r_1} \lim_{m_1,\dots,m_{r_2}\to\infty} c_{m_1,\dots,m_{r_2},n_1,\dots,n_{r_1}} = 0;$$

T2)

$$\lim_{m\to\infty} \sum_{\substack{n_1=0\\ \vdots \\ n_{r_1}=0}}^{\infty} c_{m_1,\dots,m_{r_2},n_1,\dots,n_{r_1}} = 1;$$

T3')

$$\forall m_1, \dots, m_{r_2} \exists K \sum_{\substack{n_1=0\\ \vdots \\ n_{r_1}=0}}^{\infty} \left|c_{m_1,\dots,m_{r_2},n_1,\dots,n_{r_1}}\right| < K;$$

T3")

$$\exists K, M\ \forall m_1, \dots, m_{r_2} > M \sum_{\substack{n_1=0\\ \vdots \\ n_{r_1}=0}}^{\infty} \left|c_{m_1,\dots,m_{r_2},n_1,\dots,n_{r_1}}\right| < K;$$

T4)

$$\forall n_{i_1}, \dots, n_{i_j} \forall m_1, \dots, m_{r_2}\ \exists N\ \forall n_{k_1}, \dots, n_{k_{r_1-j}} > N \quad c_{m_1,\dots,m_{r_2},n_1,\dots,n_{r_1}} = 0,$$

$$\forall n_{i_1}, \dots, n_{i_j} \exists M,N\ \forall m_1, \dots, m_{r_2} > M\ \forall n_{k_1}, \dots, n_{k_{r_1-j}} > N\ c_{m_1,\dots,m_{r_2},n_1,\dots,n_{r_1}} = 0,$$

for all $j = 1, 2, \ldots, r_1$. With j fixed, the indices $i_s, k_t \in \{1, 2, \ldots, r_1\}$, where $s = 1, 2, \ldots, j$ and $t = 1, 2, \ldots, r_1 - j$, and additionally the following conditions are fulfilled: $i_{s'} \neq i_{s''}$ for $s' \neq s''$, $k_{t'} \neq k_{t''}$ for $t' \neq t''$ and $\forall s, t\ i_s \neq k_t$.

More details and proofs of the theorems can be found in the works [2], [5] and [6].

1. Main results

The next theorem will attempt to generalize the Hardy-Littlewood theorem to the two-indicator case. First, we define Abel's method for a bounded sequence (a_{mn}) and we will show the regularity of this method.

Definition 1.1. *The double non-negative, bounded sequence (a_{mn}) is Abel summable to c if*

$$\lim_{\substack{r\to 1-\\ s\to 1-}} \sum_{m,n=0}^{\infty} a_{mn} r^m s^n (1-r)(1-s) = c, \text{ where } a_{mn} \geq 0.$$

We will prove the regularity of this method, i.e. we will show that for every non-negative and bounded double sequence (a_{mn}) we obtain: if $a_{mn} \to c$ then: $\forall r, s \in [0, 1)\ \sum\limits_{m,n=0}^{\infty} a_{mn} r^m s^n (1-r)(1-s)$ is convergent and $\lim_{\substack{r\to 1-\\ s\to 1-}} \sum_{m,n=0}^{\infty} a_{mn} r^m s^n (1-r)(1-s) = c$.

Let

$$a_{mn} \to c, \text{ to } c - \epsilon \leq a_{mn} \leq c + \epsilon \text{ for } m, n > N. \tag{1}$$

$$\begin{aligned}
\Sigma a_{mn} r^m s^n (1-r)(1-s) &\leq (1-r)(1-s) \sum_{m,n=0}^{N} r^m s^n a_{mn} + \\
+ &\sum_{\substack{m=N+1\\ n=0}}^{\substack{m=+\infty\\ n=N}} (1-r)(1-s) a_{mn} r^m s^n + \\
+ &\sum_{\substack{m=0\\ n=N+1}}^{\substack{m=+\infty\\ n=N}} (1-r)(1-s) a_{mn} r^m s^n + \sum_{m,n=N+1}^{\infty} (1-r)(1-s) a_{mn} r^m s^n.
\end{aligned} \tag{2}$$

The first term of the sum (2) converges to 0 if $r, s \to 1-$.We estimate the second part of the sum (2):

$$0 \leq (1-r)(1-s) \sum_{n=0}^{N} s^n \sum_{m=N+1}^{\infty} a_{mn} r^m \leq G(1-r)(1-s) \sum_{n=0}^{N} s^n \frac{r^{N+1}}{1-r} = G r^{N+1} (1 - s^{N+1}),$$

where $G = \sup\limits_{m,n} a_{mn}$. The second component is so that it converges to 0 if $r, s \to 1-$. The third component can be estimated in a similar way to the second one. After consideration (1) we obtain:

$$(1-r)(1-s) \sum_{m,n=N+1}^{\infty} r^m s^n a_{mn} \le (c+\epsilon)s^{N+1}r^{N+1} \to c+\epsilon.$$

We have shown that the sum on the left hand side of (2) converges to c, if $r, s \to 1-$. This means that the considered method is regular.

Theorem 1.1. *If a bounded, non-negative sequence (a_{mn}) is summable by the Abel method then it is summable by the generalized Cesáro method:*

$$\lim_{k,l\to\infty} \frac{1}{kl} \sum_{m,n=0}^{k,l} a_{mn}$$

to the same limit.

Remark 1.1. *The generalized Cesáro method is not regular.*

For example the sequence $(a_{mn})_{m,n=0}^{\infty}$, where $a_{mn} = 0$ for $m \ne 0$ and $a_{mn} = n^2$ for $m = 0$ tends to 0 as $m \to \infty, n \to \infty$. After using the generalized Cesáro method, we obtain: $v_{kl} = \frac{1}{kl} \sum\limits_{m,n=0}^{k,l} a_{mn} = \frac{(l+1)(2l+1)}{6k}$ and $\lim\limits_{k,l\to\infty} v_{kl}$ does not exist. In other words, condition T4) of Theorem 0.4 is not fulfilled.

We will precede the proof of Theorem 1.1 with the following lemma.

Lemma 1.1. *Let $f : [0,1]\times[0,1] \to R$ be a function such that for any $\epsilon > 0$ there exist functions f_1 and f_2, defined and continuous on $[0,1]\times[0,1]$, satisfying the inequality $f_1 \le f \le f_2$ and such that $\int (f_2 - f_1) < \epsilon$. Then for $a_{mn} \ge 0$ the following holds:*

$$\lim_{\substack{r\to 1-\\ s\to 1-}} (1-r)(1-s) \sum_{m,n=0}^{\infty} a_{mn} r^m s^n f(r^m, s^n) = c \cdot \int_0^1 \int_0^1 f(x,y)\, dx dy, \tag{3}$$

where

$$c = \lim_{\substack{r\to 1-\\ s\to 1-}} (1-r)(1-s) \sum_{m,n=0}^{\infty} a_{mn} r^m s^n.$$

Proof. We assume that

$$\lim_{\substack{r\to 1-\\ s\to 1-}} (1-r)(1-s) \sum_{m,n=0}^{\infty} a_{mn} r^m s^n = c. \tag{4}$$

Let's first consider the functions $f(x,y)=x^u y^v$ where $u,v\geq 0$. Then:

$$\lim_{\substack{r\to 1-\\ s\to 1-}}(1-r)(1-s)\sum_{m,n=0}^{\infty}a_{mn}r^m s^n f\left(r^m,s^n\right)=$$

$$=\lim_{\substack{r\to 1-\\ s\to 1-}}(1-r)(1-s)\sum_{m,n=0}^{\infty}a_{mn}r^m s^n\left(r^m\right)^u\left(s^n\right)^v=$$

$$=\lim_{\substack{r\to 1-\\ s\to 1-}}(1-r)(1-s)\sum_{m,n=0}^{\infty}a_{mn}\left(r^{1+u}\right)^m\left(s^{1+v}\right)^n=$$

$$=\lim_{\substack{r\to 1-\\ s\to 1-}}\frac{(1-r)(1-s)}{(1-r^{u+1})(1-s^{v+1})}\sum_{m,n=0}^{\infty}a_{mn}\left(1-r^{u+1}\right)\left(1-s^{v+1}\right)\cdot\left(r^{1+u}\right)^m\left(s^{1+v}\right)^n.$$

Let's note that

$$\lim_{\substack{r\to 1-\\ s\to 1-}}\sum_{m,n=0}^{\infty}a_{mn}\left(1-r^{u+1}\right)\left(1-s^{v+1}\right)\left(r^{1+u}\right)^m\left(s^{1+v}\right)^n=c,$$

this follows from (4). Moreover

$$\lim_{\substack{r\to 1-\\ s\to 1-}}\frac{(1-r)(1-s)}{(1-r^{u+1})(1-s^{v+1})}=\lim_{\substack{r\to 1-\\ s\to 1-}}\frac{(1-r)}{(1-r^{u+1})}\cdot\lim_{\substack{r\to 1-\\ s\to 1-}}\frac{(1-s)}{(1-s^{v+1})}=$$

$$=\lim_{r\to 1-}\frac{1}{(u+1)r^u}\cdot\lim_{s\to 1-}\frac{1}{(v+1)s^v}=\frac{1}{u+1}\cdot\frac{1}{v+1}=\int_0^1\int_0^1 x^u y^v\,dxdy.$$

Therefore

$$\lim_{\substack{r\to 1-\\ s\to 1-}}(1-r)(1-s)\sum_{m,n=0}^{\infty}a_{mn}r^m s^n f\left(r^m,s^n\right)=c\int_0^1\int_0^1 f(x,y)\,dxdy \tag{5}$$

for $f(x,y)=x^u y^v$.

Let's consider the function $g(x,y)=\sum\limits_{u,v}\alpha_{uv}x^u y^v$.

$$\lim_{\substack{r\to 1-\\ s\to 1-}}(1-r)(1-s)\sum_{m,n=0}^{\infty}a_{mn}r^m s^n\sum_{u,v}\alpha_{uv}\left(r^u\right)^m\left(s^v\right)^n=$$

$$=\sum_{u,v}\alpha_{uv}\lim_{\substack{r\to 1-\\ s\to 1-}}(1-r)(1-s)\sum_{m,n=0}^{\infty}a_{mn}r^{(1+u)m}s^{(1+v)n}=$$

$$=\sum_{u,v}\alpha_{uv}\lim_{\substack{r\to 1-\\ s\to 1-}}(1-r)(1-s)\sum_{m,n=0}^{\infty}a_{mn}r^m s^n f\left(r^m,s^n\right)=$$

$$=\sum_{u,v}\alpha_{uv}\cdot c\int_0^1\int_0^1 x^u y^v dxdy=c\cdot\int_0^1\int_0^1 g(x,y)dxdy,$$

this follows from (5) for $f(x,y)=x^u y^v$. Hence the formula (3) is true for any polynomial. We get f_1, f_2, that is:

$$f_1,f_2\in C\left([0,1]\times[0,1]\right)\text{ and }\forall x,y\in[0,1]\ f_1(x,y)\le f(x,y)\le f_2(x,y),$$

and $\int_0^1\int_0^1(f_2(x,y)-f_1(x,y))dxdy<\epsilon$, where f satisfies the assumptions of the lemma. Let P_1, P_2 – be polynomials such that

$$\forall x,y\in[0,1]\ \ |P_1(x,y)-f_1(x,y)|<\epsilon\text{ and }|P_2(x,y)-f_2(x,y)|<\epsilon.$$

Then for $a_{mn}\ge 0$ we obtain:

$$(1-r)(1-s)\textstyle\sum_{m,n=0}^{\infty}a_{mn}r^m s^n P_1\left(r^m,s^n\right)-\epsilon\le(1-r)(1-s)\sum_{m,n=0}^{\infty}a_{mn}r^m s^n f\left(r^m,s^n\right)\le$$
$$\le(1-r)(1-s)\textstyle\sum_{m,n=0}^{\infty}a_{mn}r^m s^n P_2\left(r^m,s^n\right)+\epsilon.$$

Therefore,

$$c\left(\textstyle\int_0^1\int_0^1 f(x,y)dxdy-2\epsilon\right)\le$$
$$\le c\left(\textstyle\int_0^1\int_0^1 P_1(x,y)dxdy-\epsilon\right)\le\liminf\limits_{\substack{r\to 1-\\ s\to 1-}}\textstyle\sum_{n,m=0}^{\infty}a_{mn}r^m s^n f\left(r^m,s^n\right)$$

and

$$c\left(\textstyle\int_0^1\int_0^1 f(x,y)dxdy-2\epsilon\right)\ge$$
$$\ge c\left(\textstyle\int_0^1\int_0^1 P_2(x,y)dxdy+\epsilon\right)\ge\limsup\limits_{\substack{r\to 1-\\ s\to 1-}}\textstyle\sum_{m,n=0}^{\infty}a_{mn}r^m s^n f\left(r^m,s^n\right).$$

Taking into account the arbitrariness of ϵ, we get:

$$\lim_{r\to 1-,s\to 1-}\sum_{m,n=0}^{\infty}a_{mn}r^m s^n f\left(r^m,s^n\right)=c\cdot\int_0^1\int_0^1 f(x,y)dxdy,$$

which completes the proof of the Lemma 1.1. □

Proof of the Theorem 1.1

Proof. We assume that the sequence (a_{mn}) is summable by the Abel method to the value c. Let's define:

$$h(x,y)=\begin{cases}\dfrac{1}{xy} & \text{for } x\geq\dfrac{1}{e} \text{ and } y\geq\dfrac{1}{e},\\ 0 & \text{for others } x,y\end{cases}.$$

The function h satisfies the assumptions of the Lemma 1.1.

$$\lim_{\substack{r\to1-\\ s\to1-}}\sum_{m,n=0}^{\infty}(1-r)(1-s)a_{mn}r^m s^n h\left(r^m,s^n\right)=\lim_{\substack{r\to1-\\ s\to1-}}(1-r)(1-s)\sum_{\substack{m:r^m\geq\frac{1}{e}\\ n:s^n\geq\frac{1}{e}}}a_{mn}r^m s^n\frac{1}{r^m s^n}=$$

$$=\lim_{\substack{r\to1-\\ s\to1-}}(1-r)(1-s)\sum_{\substack{m:r^m\geq\frac{1}{e}\\ n:s^n\geq\frac{1}{e}}}a_{mn}. \quad (6)$$

After consideration the Lemma 1.1 we obtain

$$\lim_{\substack{r\to1-\\ s\to1-}}\sum_{m,n=0}^{\infty}(1-r)(1-s)a_{mn}r^m s^n h\left(r^m,s^n\right)=c\int_0^1\int_0^1 h(x,y)dxdy=c\left(\int_{1/e}^1 1/xdx\right)^2=c. \quad (7)$$

Therefore

$$\lim_{\substack{r\to1-\\ s\to1-}}(1-r)(1-s)\sum_{\substack{m:r^m\geq\frac{1}{e}\\ n:s^n\geq\frac{1}{e}}}a_{mn}=c, \quad (8)$$

this follows from (6) and (7). If $r_k=\left(\frac{1}{e}\right)^{\frac{1}{k}}$ and $s_l=\left(\frac{1}{e}\right)^{\frac{1}{l}}$ then $\lim\limits_{k,l\to\infty}(r_k,s_l)=(1-,1-)$ and:

$$\lim_{\substack{k\to\infty\\ l\to\infty}}k\left(1-\left(\frac{1}{e}\right)^{\frac{1}{k}}\right)l\left(1-\left(\frac{1}{e}\right)^{\frac{1}{l}}\right)\sum_{\substack{m\leq k\\ n\leq l}}\frac{a_{mn}}{kl}=$$

$$=\lim_{k\to\infty}k\left(1-\left(\frac{1}{e}\right)^{\frac{1}{k}}\right)\lim_{l\to\infty}l\left(1-\left(\frac{1}{e}\right)^{\frac{1}{l}}\right)\lim_{k,l\to\infty}\frac{1}{kl}\sum_{m,n=0}^{k,l}a_{mn}=$$

$$=\lim_{k,l\to\infty}\frac{1}{kl}\sum_{m,n=0}^{\substack{m=k\\ n=l}}a_{mn}=c,$$

this follows from (8). □

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